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p-adic spectral measure

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Fuglede conjecture

Conjecture(1974)

Ω is a tile of $\mathbb{R}^d$ if and only if it is a spectral set.

- It is proved under the extra condition: T or Λ is a lattice. (Fuglede 1974)
- Not true for $d \geq 3$ on both direction:
  $d \geq 5$, a spectral set but not a tile.
  (Tao 2003)
  $d = 3, 4$.
  (Farkas, Gy, Matolsci and Kolountzakis 2006).
- It remains open for $d = 1, 2$.
- True for convex sets. (Lev-Matolsci, 2022)

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LCA groups

- Let G be a **locally compact abelian group**. Let m is the Haar measure.
- Let $\widehat{G}$ be the dual group of G.

$$\widehat{G} := \{\chi : G \to S^1 | \text{continous group homomorphism}\}$$

- Examples: $\widehat{\mathbb{R}^d} = \mathbb{R}^d$, $\widehat{\mathbb{R}/\mathbb{Z}} = \mathbb{Z}$.
- Let $\Omega \subset G$ be a Borel set with $0 < m(\Omega) < +\infty$.
- We say that the set Ω **tiles** G by translations, if there exists a set $T \subset G$ such that $\{\Omega + t : t \in T\}$ forms a partition a.e. of G.
- The set Ω is said to be **spectral** if there exists a set $\Lambda \in \widehat{G}$ such that Λ forms an orthogonal basis of $L^2(\Omega)$.

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Conjecture

A Borel set Ω is a tile if and only if it is a spectral set.

- S-$T(G)$, respectively T-$S(G)$, if the Spectral$\Rightarrow$Tile direction, respectively Tile$\Rightarrow$Spectral, holds in G;
  (i) T-$S(\mathbb{R}) \Leftrightarrow T$-$S(\mathbb{Z}) \Leftrightarrow T$-$S(\mathbb{Z}/m\mathbb{Z})$ for all $m \in \mathbb{N}$
  (ii) S-$T(\mathbb{R}) \Rightarrow S$-$T(\mathbb{Z}) \Rightarrow S$-$T(\mathbb{Z}/m\mathbb{Z})$ for all $m \in \mathbb{N}$
  (Dutkay, Lai 2013)

Fuglede conjecture in LCA groups

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Fuglede conjecture on local field

Local field	What we know
$\mathbb{R}$	still open
$\mathbb{C} \simeq \mathbb{R}^2$	still open
$\mathbb{F}_q((T))^d, q = p^n$	fail
$\mathbb{Q}_p^d$	$\mathbb{Q}_p$ holds (Fan–Fan–Liao–S, 2019) $\mathbb{Q}_p^5$ $(p \equiv 1 \mod 4)$, $\mathbb{Q}_p^4$ $(p \equiv 3 \mod 4)$, $\mathbb{Q}_2^3$ fail $\mathbb{Q}_p^2$ still open

Why $\mathbb{Q}_p$?

- A bridge: finite field $\to$ p-adic field $\to$ field of real numbers.
- Examples: Kakeya conjecture
  - Kakeya posed the needle problem (1917): Kakeya set.
  - Wolff (1999) suggested a Kakeya conjecture over finite fields, proved by Dvir (2009).
  - Ellenberg–Oberlin–Tao (2010) suggested analogues of the Kakeya conjecture over topological fields whose metrics capture the multiple scales property of the Euclidean metric of $\mathbb{R}^n$: the finite extensions of $\mathbb{R}$, $\mathbb{Q}_p$ and $\mathbb{F}_q((T))^d$, $q = p^n$.
  - Arsovski (JAMS, 2024): the p-adic Kakeya conjecture holds.
  - Wang-Zahl (preprint, 2025): Kakeya conjecture on $\mathbb{R}^3$ holds.

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Spectral measures on LCA

- A Borel measure ν on a LCA groups G is a **spectral measure** if $\exists \Lambda \subset \widehat{G}$ such that Λ is an orthogonal basis of $L^2(\nu)$.
- **Law of pure type**: a spectral measure is
  - either *absolutely continuous* (w.r.t. Haar measure),
  - *discrete*,
  - or *singular continuous*.

  (Łaba-Wang, 2006 & He-Lai-Lau, 2013)
- A absolutely continuous spectral measure is of the form $1_\Omega d\mu$ for some Borel set Ω. (Dutkay-Lai, 2014)
- A discrete spectral measure is of the form $\sum_{c \in C} a \cdot \delta_c$ for some set $C \subset G$ and $a > 0$. (Łaba-Wang, 2006)

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Examples

- In finite groups, spectral measure $\Leftrightarrow$ spectral set.
- In $\mathbb{R}$,
  $\frac{1}{3}$-Cantor measure $\rightarrow$ non-spectral measure;
  $\frac{1}{4}$-Cantor measure $\rightarrow$ spectral measure.

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Main result

Theorem (S, 2023)

A Borel measure on $\mathbb{Q}_p$ is a spectral measure if and only it is p-homogeneous, i.e. the weak-$*$ limit of discrete spectral measures.

- Question: Same phenomenon in $\mathbb{R}$?

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The field of p-adic numbers

- Let p be a prime and $p \geq 2$.
- Any nonzero number $r \in \mathbb{Q}$ can be written as $r = p^v \frac{a}{b}$ where $v, a, b \in \mathbb{Z}$ and $(p, a) = 1$ and $(p, b) = 1$.
- Define $v_p(r) = v$ and $|r|_p = p^{-v_p(r)}$ for $r \neq 0$ and $|0|_p = 0$.
- $|\cdot|_p$ is a non-Archimedean absolute value. That means
  (i) $|r|_p \geq 0$ with equality only for $r = 0$;
  (ii) $|rs|_p = |r|_p |s|_p$;
  (iii) $|r + s|_p \leq \max\{|r|_p, |s|_p\}$.
- The field of p-adic numbers $\mathbb{Q}_p$ is the completion of $\mathbb{Q}$ under $|\cdot|_p$.

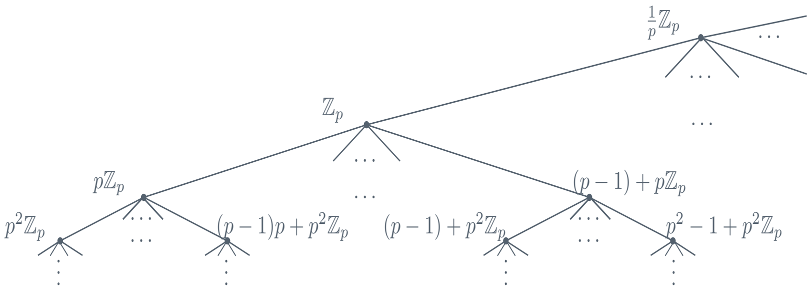
The field of p-adic numbers

- An element of $\mathbb{Q}_p$ is of the form

$$\sum_{n=N}^{\infty} a_n p^n \qquad \text{where} N \in \mathbb{Z}, a_n \in \{0, 1, \cdots, p-1\}.$$

- $\mathbb{Z}_p = \{x \in \mathbb{Q}_p : |x|_p \leq 1\}$ is the ring of p-adic integers.
- m is the Haar measure with $m(\mathbb{Z}_p) = 1$.

Tree model of $\mathbb{Q}_p$



The dual group of $\mathbb{Q}_p$

- The fractional part of $x = \sum_{n=N}^{\infty} a_n p^n$ is $\{x\} = \sum_{n=N}^{-1} a_n p^n$.
- An example of a non-trivial additive character is

$$\chi(x) = e^{2\pi i \{x\}}.$$

- From this character we can get all characters χ_y of $\mathbb{Q}_p$, defined by $\chi_y(x) = \chi(yx)$.
- $\widehat{\mathbb{Q}_p} = \mathbb{Q}_p$.

Fourier transform on $\mathbb{Q}_p$

- The Fourier transformation of $f \in L^1(\mathbb{Q}_p)$ is defined to be

$$\widehat{f}(y) = \int_{\mathbb{Q}_p} f(x)\overline{\chi_y(x)}dx \quad (y \in \widehat{\mathbb{Q}_p} \simeq \mathbb{Q}_p).$$

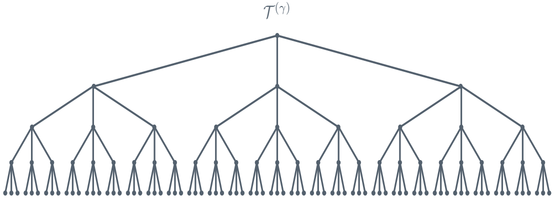
Lemma

Let $\gamma \in \mathbb{Z}$ be an integer.
(a) We have $\widehat{1_{B(0,p^\gamma)}}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi)$ for all $\xi \in \mathbb{Q}_p$.
(b) If $\Omega = \bigsqcup_j B(c_j, p^\gamma)$ is a finite union of disjoint balls of the same size, then

$$\widehat{1_\Omega}(\xi) = p^\gamma 1_{B(0,p^{-\gamma})}(\xi) \sum_j \chi(-c_j\xi).$$

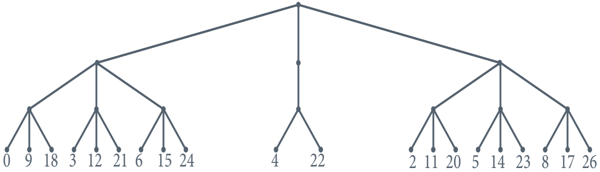
$\mathbb{Z}/p^\gamma\mathbb{Z}$ as a Tree

- We identify $\mathbb{Z}/p^\gamma\mathbb{Z} = \{0, 1, \ldots, p^\gamma - 1\}$ with the following finite tree, denoted by $\mathcal{T}^{(\gamma)}$.
- Example: $\mathbb{Z}/3^4\mathbb{Z}$



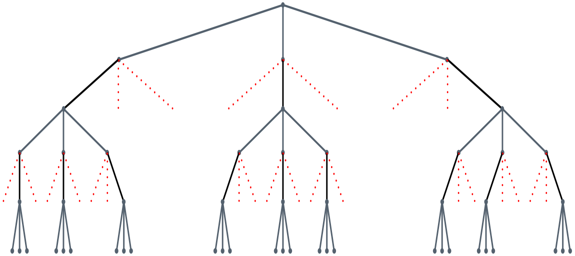
Compact open set as a Tree

- Each subset $C \subset \mathbb{Z}/p^\gamma\mathbb{Z}$ will determine a subtree of $\mathcal{T}^{(\gamma)}$, denoted by $\mathcal{T}_C$, which consists of the paths from the root to the points in C.
- Example:
  $C = \{0, 2, 3, 4, 5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26\}$
  as a subset of $\mathbb{Z}/3^4\mathbb{Z}$:



p-homogenous tree

- The tree $\mathcal{T}_C$ is called p**-homogenous** if the number of descendants the vertices at the same level is same and the number is either 1 or p.
- Example: $p = 3, \gamma = 2$, $C = \{0, 4, 8, 9, 13, 17, 18, 22, 26\}$.



Idea of the proof: Bruhat-Schwartz distributions

- Bruhat-Schwartz test functions $\mathcal{D}$: uniformly locally constant functions of compact support.
- Bruhat-Schwartz distribution $\mathcal{D}'$: a continuous linear functional on the space of Bruhat-Schwartz test functions.

Proposition

Let $f, g \in \mathcal{D}'$. If the product $f \cdot g$ is well defined and equal to h, then we have

$$\mathcal{Z}_h \cap \mathcal{S}_f \subset \mathcal{Z}_g.$$

Open problems

Question

local field: all tiles are spectral sets? Non-spectral set tiles?

Question

$\mathbb{Q}_p^d$: all tiles are spectral sets? Non-spectral set tiles?

Question

$(\mathbb{Z}/p^n\mathbb{Z})^d$: all tiles are spectral sets? Non-spectral set tiles?

- Non-spectral set tiles: finite abelian group with at least two prime factors.

Spectral set as product structure

Question

Suppose $A \times B \subset \mathbb{Q}_p^n \times \mathbb{Q}_p^m$ is a spectral set. Is it true that both A and B are spectral sets?

- $A \times B \subset \mathbb{Z}/p^n\mathbb{Z} \times G$ is a spectral set with G finite abelian group , then both A and B are spectral sets. (Fan-S-Zhang, 2025+)
- $A \times B \subset \mathbb{R} \times \mathbb{R}$ is a spectral set. Is it true that both A and B are spectral sets?
  - A is an interval. (Greenfeld-Lev, 2016)
  - A is a union of two intervals. (Kolountzakis, 2016)

Spectral measure as product structure

Question

Suppose $\mu \times \nu \in \mathcal{P}(\mathbb{Q}_p^n) \times \mathcal{P}(\mathbb{Q}_p^m)$ is a spectral measure. Is it true that both μ and ν are spectral measures?

- $\mu \times \nu \in \mathcal{P}(\mathbb{R}) \times \mathcal{P}(\mathbb{R})$ is a spectral measure. Is it true that both μ and ν are spectral measures?
- Law of pure type?
- Under certain assumption: μ is the $\frac{1}{4}$-Cantor set or λ-Bernoulli convolution?

Weak tiling

Question

$\mathbb{Q}_p$: weak tiles are tiles/spectral sets?

- $\mathbb{Z}/p^n\mathbb{Z}$: weak tiles are tiles and spectral sets. (S, 2025+)
- For $\mathbb{Z}/N\mathbb{Z}$, $\mathbb{Z}$ or $\mathbb{R}$?

Frame

Question

Can we classify frame spectral measures on $\mathbb{Q}_p$?

- Cantor measure in $\mathbb{Q}_p$?

Thanks for your attention!